Multi-time correlations in quantized toral automorphisms

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Abstract

The long time asymptotics of multi-time correlation functions of relaxing quantum mechanical systems can be conveniently studied by means of free-products of suitable C*-algebras and of states on these free products given by multiple temporal averages. In this paper, we study the distribution law of fluctuations of temporal averages in a class of quantized toral automorphisms.

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1 Introduction

Recently, it has been suggested [BF, ABDF] that the statistics of long time asymptotics of multi-time correlation functions of quantum dynamical systems might be used to associate particular probability distribution laws with the variety of phenomena commonly going under the name of quantum chaos [CC]. The proposed setting is that of a free-product of copies of the algebra of obervables of the system, equipped with a state obtained by a suitable multiple time-average.

Given an invariant state and expectations of observables at different times, with a same time possibly appearing more than once, one may try to express such expectations in terms of time-ordered correlation functions. The intrinsic complexity of the dynamics might render this task very hard indeed and is likely to generate a more or less unwieldy proliferation of commutators. We will in particular address the situation where differences between different times become large. In the particular case of strong time-asymptotic commutativity one can easily group observables at equal, largely separated, times.

In general, the idea is to leave aside any attempt at simplifying multi-time correlation functions by expressing them in terms of ordered correlation functions, but rather to regard them as elements of a free-product algebra whose expectations are obtained by averaging with respect to all different times. It is plausible that the chaotic features of the dynamics, if any, might leave their imprinting on the statistical properties of the asymptotic state.

Ultimately, the testing ground of the scheme of above should consist of those quantum systems, typically finite dimensional, that tend to classically chaotic systems when the dimension of the associated Hilbert spaces increases to infinity [BV, BBTV, CC, D, HKS]. This involves also an appropriate rescaling of the time. In this paper, we analyse a family of toy models, namely the hyperbolic toral automorphisms quantized as in [BNS]. These models are infinite dimensional quantum systems arising via a non-commutative deformation of the algebra of continuous functions on the two-dimensional torus. The quantum algebra is equipped with the tracial state and endowed with a dynamics such that the GNS time-evolutor coincides with the classical Koopman-von Neumann unitary operator. The absolute continuity of the spectrum of the quantum evolution guarantees those clustering properties which are ruled out in the commonly studied classically chaotic quantum systems which usually have discrete spectra and quasi-periodic time-behaviour.

The mixing properties depend on the value of a certain deformation parameter θ and one can distinguish different randomness conditions that lead, via a central limit theorem, to a variety of statistics of fluctuations ranging from the Gaussian distribution to Wigner's semi-circle law.

2 Statistics of multi-time correlation functions

We briefly resume the approach of [ABDF]. $(\mathfrak{A}, \Theta, \phi)$ will denote a discrete time dynamical system where

- $\mathfrak{A}$ is a unital C*-algebra
- $\Theta = \Big\{\Theta_t \,|\, t \in \mathbf{Z}\Big\}$ is a discrete dynamical group of automorphisms of $\mathfrak{A}$ such that $X(t) := \Theta_t(X)$ represents the operator X evolved up to time t
- ϕ is a Θ-invariant state on $\mathfrak{A}$: $\phi \circ \Theta = \phi$.

We shall consider multi-time correlation functions of the form

$$\mathbf{t} \mapsto \phi\Big(X^{(1)}(t_{\nu(1)})X^{(2)}(t_{\nu(2)}) \cdots X^{(n)}(t_{\nu(n)})\Big) \ , \tag{2.1}$$

where the $X^{(j)}(t_{\nu(j)})$ are operators at times $t_{\nu(j)}$ in $\mathfrak{A}$, $\mathbf{t} = \{t_1, t_2, \ldots\} \in \mathbf{Z}^{\mathbf{N}_0}$ and ν maps $\{1, 2, \ldots, n\}$ into $\mathbf{N}_0$. Two consecutive time-indices will always be considered different, otherwise, if, say $\nu(j) = \nu(j+1) = p$, then we write $X^{(j)}(t_{\nu(j)})X^{(j+1)}(t_{\nu(j+1)}) = \Big(X^{(j)}X^{(j+1)}\Big)(t_p)$. On the other hand, we allow $\nu(j)$ to be equal to one or more of the $\nu(\ell)$ when $\ell \neq j \pm 1$.

As outlined in the introduction, while in quantum statistical mechanics it is commonly expected that an expression as in (2.1) can always be reordered by bringing together operators at equal times, in this paper we would like to consider dynamical situations where the commutation relations between operators largely separated in time are of almost no use. Then, the natural algebraic structure to consider is that of a countable free-product $\mathfrak{A}_\infty = \star_{i \in \mathbf{N}_0} \mathfrak{A}_i$ of copies of $\mathfrak{A}$, which is the universal C*-algebra generated by an identity element $\mathbb{1}$ and by "words" $w = X^{(1)}_{\nu(1)} X^{(2)}_{\nu(2)} \cdots X^{(n)}_{\nu(n)}$ that consist of concatenations of "letters" $X^{(j)} \in \mathfrak{A}$. The subscript $\nu(j)$ in $X^{(j)}_{\nu(j)}$ refers to which copy of $\mathfrak{A}$ the letter $X^{(j)}$ belongs. Concatenation, together with simplification rules, defines the product of words. More specifically, the rules for handling words are: for X, $Y \in \mathfrak{A}$, $\lambda \in \mathbf{C}$, $j \in \mathbf{N}_0$ and w, w' two generic words

$$w \mathbb{1}_j w' = w w' \tag{2.2.a}$$

$$w\big(X_j + \lambda Y_j\big)w' = wX_jw' + \lambda wY_jw' \tag{2.2.b}$$

$$wX_jY_jw' = w(XY)_jw' \ . \tag{2.2.c}$$

Notice that the product XY in (2.2.c) is not concatenation, but rather the usual operator product in the algebra $\mathfrak{A}$. Moreover, the adjoint w^* of a word $w = X^{(1)}_{\nu(1)} X^{(2)}_{\nu(2)} \cdots X^{(n)}_{\nu(n)}$ equals $\big(X^{(n)*}\big)_{\nu(n)} \big(X^{(n-1)*}\big)_{\nu(n-1)} \cdots \big(X^{(1)*}\big)_{\nu(1)}$.

We shall refer to $\mathfrak{A}_\infty$ as to the asymptotic free algebra and equip it with an asymptotic state ϕ_∞ as follows. With the notation

$$\mathrm{Avg}\Big(t\mapsto f(t)\Big)=\lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T-1}f(t)\ , \tag{2.3}$$

ν any map from $\{1,2,\ldots,n\}$ into $\mathbf{N}_0$ as above, we assume that the multiple average

$$\mathbf{Avg}\Big(\mathbf{t}\mapsto\phi\Big(X^{(1)}(t_{\nu(1)})X^{(2)}(t_{\nu(2)})\cdots X^{(n)}(t_{\nu(n)})\Big)\Big):=\mathrm{Avg}\Big(t_n\mapsto\cdots\mathrm{Avg}\Big(t_2\mapsto \mathrm{Avg}\Big(t_1\mapsto\phi\Big(X^{(1)}(t_{\nu(1)})X^{(2)}(t_{\nu(2)})\cdots X^{(n)}(t_{\nu(n)})\Big)\Big)\Big)\cdots\Big)$$

with $n=1,2,...$, exists. Then, we define a linear functional ϕ_∞ on $\mathfrak{A}_\infty$ by linearly extending the map on elementary words $X^{(1)}_{\nu(1)}X^{(2)}_{\nu(2)}\cdots X^{(n)}_{\nu(n)}$,

$$\phi_\infty\Big(X^{(1)}_{\nu(1)}X^{(2)}_{\nu(2)}\cdots X^{(n)}_{\nu(n)}\Big):=\mathbf{Avg}\Big(\mathbf{t}\mapsto\phi\Big(X^{(1)}(t_{\nu(1)})X^{(2)}(t_{\nu(2)})\cdots X^{(n)}(t_{\nu(n)})\Big)\Big)\ . \tag{2.4}$$

The linear functional ϕ_∞ is such that it does not depend on the insertion of identities, namely

$$\mathrm{Avg}\Big(t_3\mapsto\mathrm{Avg}\Big(t_2\mapsto\mathrm{Avg}\Big(t_1\mapsto\phi\Big(\mathbb{1}(t_1)\mathbb{1}(t_2)X(t_3)Y(t_1)Z(t_3)\Big)\Big)\Big)\Big)= \mathrm{Avg}\Big(t_3\mapsto\mathrm{Avg}\Big(t_1\mapsto\phi\Big(X(t_3)Y(t_1)Z(t_3)\Big)\Big)\Big)\ .$$

Thus, $\phi_\infty\Big(X_3Y_1Z_3\Big)=\phi_\infty\Big(X_2Y_1Z_2\Big)$ by relabelling the dummy summation indices. Moreover, the linear functional ϕ_∞ is positive and one has the following general result:

Proposition 1. *The functional ϕ_∞ defined in (2.4) extends to a state on $\mathfrak{A}_\infty$. Furthermore,*

$$\phi_\infty\circ\mathbf{\Theta_s}=\phi_\infty\qquad\textit{and}\qquad\phi_\infty\circ\alpha_\theta=\phi_\infty\ ,$$

where $\mathbf{\Theta_s}$ satisfies $\mathbf{\Theta_s}(X_j):=\big(\Theta_{s_j}(X)\big)_j$ for $j\in\mathbf{N}_0$ and $X\in\mathfrak{A}$, whereas θ is any order preserving injective transformation of $\mathbf{N}_0$ and α_θ is the $*$-homomorphism of $\mathfrak{A}_\infty$ determined by $\alpha_\theta(X_j):=X_{\theta(j)}$.

We shall say that the asymptotic state ϕ_∞ is permutation invariant when

$$\phi_\infty\Big(X^{(1)}_{\nu(1)}X^{(2)}_{\nu(2)}\cdots X^{(n)}_{\nu(n)}\Big)=\phi_\infty\Big(X^{(1)}_{\pi\circ\nu(1)}X^{(2)}_{\pi\circ\nu(2)}\cdots X^{(n)}_{\pi\circ\nu(n)}\Big)\qquad\forall\, X^{(j)}\in\mathfrak{A}\ , \tag{2.5}$$

where $\pi:\mathbf{N}_0\mapsto\mathbf{N}_0$ is any bijection.

3 Clustering properties and fluctuations

The clustering properties of the dynamical system $(\mathfrak{A}, \Theta, \phi)$ determine much of the structure of the asymptotic state ϕ_∞ and hence of the statistics on the asymptotic free algebra $\mathfrak{A}_\infty$ associated with $(\mathfrak{A}, \Theta, \phi)$.

Typically, two degrees of mixing are to be distinguished in quantum systems, called weak and strong clustering. They correspond to

$$\lim_{t\to\infty} \phi(X\,Y(t)\,Z) = \phi(X\,Z)\,\phi(Y), \qquad X,\,Y,\,Z \in \mathfrak{A}\ , \tag{3.1}$$

respectively

$$\lim_{t\to\infty} \phi(X\,Y(t)\,Z\,S(t)\,T) = \phi(X\,Z\,T)\,\phi(Y\,S), \qquad X,\,Y,\,Z,\,S,\,T \in \mathfrak{A}\ . \tag{3.2}$$

The latter property, if it holds, is equivalent to hyper-clustering, that is to [NT1, ABDF]

$$\lim_{\inf|t_i - t_j|\to\infty} \phi\Big(X^{(1)}(t_{\nu(1)})X^{(2)}(t_{\nu(2)})\cdots X^{(n)}(t_{\nu(n)})\Big) = \prod_j \phi\Big(\overrightarrow{\prod}_{\kappa\in\nu^{-1}(j)} X^{(\kappa)}\Big)\ , \tag{3.3}$$

where the limit is taken in such a way that all times and the differences between the different ones go to infinity. It is hyper-clustering that allows the reordering of multi-time correlation functions (2.1) with repeated times, when different times become largely separated. On the level of the asymptotic state ϕ_∞, strong (equivalently hyper-) clustering leads to:

Proposition 2. *Let $(\mathfrak{A}, \Theta, \phi)$ be strongly clustering. Then, the asymptotic state ϕ_∞ defined on $\mathfrak{A}_\infty$ by (2.4) is permutation invariant.*

Essentially, if (3.2) holds, then using hyper-clustering, the order in which the single time-averages in (2.4) are performed does not matter for the averages coincide with the time-limits (3.3) of the multi-time correlation functions. Moreover, from $\phi\circ\Theta = \phi$ it follows that $\phi_\infty(X_j) = \phi(X)$ for any $j \in \mathbf{N}_0$ and $X \in \mathfrak{A}$, whence

$$\phi_\infty\Big(X^{(1)}_{\nu(1)}X^{(2)}_{\nu(2)}\cdots X^{(n)}_{\nu(n)}\Big) = \prod_j \phi\Big(\overrightarrow{\prod}_{k\in\nu^{-1}(j)} X^{(k)}\Big)\ . \tag{3.4}$$

The structure of the expectations on $\mathfrak{A}_\infty$ calculated with respect to the asymptotic state ϕ_∞ is compatible with imposing for $j \neq k$ the commutation relations $[\mathfrak{A}_j, \mathfrak{A}_k] = 0$ on $\mathfrak{A}_\infty$ and corresponds to the usual notion of commutative independence of random variables.

It is well-known that the notion of independence embodied in (3.4) is incompatible with that of free independence [VDN] which asks that the correlation functions of a state ψ on a free product $\star_j\mathfrak{B}_j$ of C*-algebras $\mathfrak{B}_j$ satisfy

$$\psi\Big(X^{(1)}_{j_1}X^{(2)}_{j_2}\cdots X^{(n)}_{j_n}\Big) = 0 \tag{3.5}$$

whenever $j_k \neq j_{k+1}$ and $X_{j_k}^{(k)}$ is centred (i.e. $\psi\Big(X_{j_k}^{(k)}\Big) = 0$) for all k.

The two notions of independence of above are somehow extreme. Many other possibilities for the structure of ϕ_∞ may arise from the dynamical properties of $(\mathfrak{A}, \Theta, \phi)$ and we shall try to expose some of them by looking at limits of the form

$$\lim_{N\to\infty} \phi_\infty\Big(F_N(X^{(1)})F_N(X^{(2)})\cdots F_N(X^{(r)})\Big) \ , \tag{3.6}$$

and establishing a central limit theorem for the local fluctuations $F_N(X)$.

Definition 1. *Let N be a natural number and $X \in \mathfrak{A}$. A local fluctuation $F_N(X)$ is the following element of $\mathfrak{A}_\infty$*

$$F_N(X) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\Big(X_i - \phi(X)\mathbb{1}\Big) \ . \tag{3.7}$$

By adapting an argument in [SvW], it can be shown that the cluster condition

$$\lim_{\inf|t_i - t_j|\to\infty} \phi\Big(Z^{(1)}(t_{\nu(1)})\cdots Z^{(j)}(t_{\nu(j)})YZ^{(j+1)}(t_{\nu(j+1)})\cdots Z^{(n)}(t_{\nu(n)})\Big) = 0 \ , \tag{3.8}$$

for all centred $Y \in \mathfrak{A}$ and $\nu : \{1, 2, \ldots, n\} \mapsto \mathbf{N}_0$, which is stronger than weak clustering (3.1), but weaker than strong clustering (3.2), is sufficient to ensure that only moments of even order contribute to the limit joint distribution of fluctuations.

Proposition 3. *Let us assume that (3.8) holds in $(\mathfrak{A}, \Theta, \phi)$. Then, with ϕ_∞ defined by (2.4) and $X^{(1)}, \ldots, X^{(r)}$ in $\mathfrak{A}$ centred observables,*

$$\lim_{N\to\infty} \phi_\infty\Big(F_N(X^{(1)})F_N(X^{(2)})\cdots F_N(X^{(r)})\Big) = \begin{cases} 0 & r = 2n+1 \\ \dfrac{1}{n!}\sum_{\nu}^{(2)} \phi_\infty\Big(X_{\nu(1)}^{(1)}\cdots X_{\nu(2n)}^{(2n)}\Big) & r = 2n. \end{cases}$$

$\sum_{\nu}^{(2)}$ means that we have to sum over all partitions ν of $\{1, 2, \ldots, 2n\}$ into pairs $\big((\alpha_1, \beta_1),$ $(\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\big)$ i.e. we choose sites $\alpha_j < \beta_j$ such that $\nu(\alpha_j) = \nu(\beta_j) = j$ with j running from 1 to n.

As an immediate consequence, we also have

Corollary 1. *If the asymptotic state ϕ_∞ is permutation invariant in the sense of (2.5), and $X^{(1)}, \ldots, X^{(r)}$ are centred observables of $\mathfrak{A}$, then*

$$\lim_{N\to\infty} \phi_\infty\Big(F_N(X^{(1)})F_N(X^{(2)})\cdots F_N(X^{(r)})\Big) = \begin{cases} 0 & r = 2n+1 \\ \sum_{\nu,\,\mathrm{ord}}^{(2)} \phi_\infty\Big(X_{\nu(1)}^{(1)}\cdots X_{\nu(2n)}^{(2n)}\Big) & r = 2n \ , \end{cases}$$

where $\sum_{\nu,\,\mathrm{ord}}^{(2)}$ *means that the sum is over all* ordered *pair partitions* $\nu = \Big((\alpha_1,\beta_1),(\alpha_2,\beta_2),\,\ldots,(\alpha_n,\beta_n)\Big)$ *of* $\{1,2,\ldots,2n\}$*, i.e. we choose sites* $\alpha_j < \beta_j$ *such that* $\nu(\alpha_j) = \nu(\beta_j) = j$ *with* j *running from 1 to* n *and* $\alpha_1 < \alpha_2 < \cdots < \alpha_n$.

Corollary 2. *If the dynamical system* $(\mathfrak{A},\Theta,\phi)$ *is strongly clustering, then the fluctuations* $F_N(X)$ *of observables* $X \in \mathfrak{A}$ *such that* $\phi(X) = 0$ *and* $\phi(X^2) = \sigma^2$ *tend to Gaussian random variables with zero mean and variance* σ.

A crossing occurs in a given a pair partition ν of $\{1,2,\ldots,2n\}$ into n pairs (α_j,β_j), with $\alpha_j < \beta_j$, when

$$\alpha_j < \alpha_k < \beta_j < \beta_k \quad \text{for some} \quad j,k \in \{1,2,\ldots,n\}\ . \tag{3.10}$$

Denoting by $c(\nu)$ the number of crossings in ν, we say that ν is non-crossing if $c(\nu) = 0$. If ν is a non-crossing pair partition of $\{1,2,\ldots,2n\}$, then its pairs (α_j,β_j), $j = 1,2,\ldots,n$, are nested, that is if $\alpha_j < \alpha_k < \beta_j$ for some j,k, then also $\beta_k < \beta_j$. According to expectations associated with pair partitions, one has various types of generalized Brownian motions [BKS,BS,S,SvW,vLM], that can be characterized by the contribution $-1 \le q \le 1$ of each crossing

$$\phi_\infty\Big(X^{(1)}_{\nu(1)}\cdots X^{(2n)}_{\nu(2n)}\Big) = q^{c(\nu)}\prod_{k=1}^{n}\phi\Big(X^{(\alpha_k)}X^{(\beta_k)}\Big)\ . \tag{3.11}$$

For $q = \pm 1$ the Brownian motion is termed Bosonic, respectively Fermionic, while for $q = 0$ the q-deformed Brownian motion is called Free [VDN] and to it only non-crossing pair partitions do contribute.

It turns out that if, besides the cluster condition (3.8), it also holds that, for any time independent choice of observables A and C and centred observables X, Y, B,

$$\mathrm{Avg}\Big(t \mapsto \phi\Big(AX(t)BY(t)C\Big)\Big) = 0\ , \tag{3.12}$$

then a free statistics for the fluctuations of temporal averages emerges. Namely,

Proposition 4. *Let* $(\mathfrak{A},\Theta,\phi)$ *satisfy conditions (3.8) and (3.12). Then, with* ϕ_∞ *the asymptotic state on* $\mathfrak{A}_\infty$ *defined by (2.4) and* $X^{(1)},\ldots,X^{(2n)}$ *centred observables in* $\mathfrak{A}$,

$$\lim_{N\to\infty}\phi_\infty\Big(F_N(X^{(1)})F_N(X^{(2)})\cdots F_N(X^{(2n)})\Big) = \frac{1}{n!}\sum_{\nu}^{(2)}\delta_{0,c(\nu)}\prod_{k=1}^{n}\phi\Big(X^{(\alpha_k)}X^{(\beta_k)}\Big)\ ,$$

where the sum extends over all non-crossing pair partitions $\nu = \Big((\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)\Big)$ *of* $\{1,2,\ldots,2n\}$ *and* $c(\nu)$ *denotes the number of crossings in* ν*. If the asymptotic state* ϕ_∞ *is also permutation invariant, then*

$$\lim_{N\to\infty}\phi_\infty\Big(F_N(X^{(1)})F_N(X^{(2)})\cdots F_N(X^{(2n)})\Big) = \sum_{\nu}^{(2)}\delta_{0,c(\nu)}\prod_{k=1}^{n}\phi\Big(X^{(\alpha_k)}X^{(\beta_k)}\Big)\ ,$$

where the sum now extends over all ordered and non-crossing pair partitions ν of $\{1, 2, \dots, 2n\}$.

Corollary 3. *If the dynamical system $(\mathfrak{A}, \Theta, \phi)$ satisfies conditions (3.8), (3.1) and the asymptotic state ϕ_∞ is permutation invariant as in (2.5), then the fluctuations $F_N(X)$ of observables $X \in \mathfrak{A}$ such that $\phi(X) = 0$ and $\phi(X^2) = \sigma^2$ tend to semicircularly distributed random variables with zero mean and variance σ.*

4 Quantized automorphisms of the torus

We shall consider discrete time dynamical systems $(\mathfrak{A}, \Theta, \phi)$ constructed as follows. Let u and v be two unitary operators satisfying the commutation relations

$$u\,v = \mathrm{e}^{2i\pi\theta}\ v\,u\ ,$$

where $0 \leq \theta < 1$ will be referred to as the deformation parameter. Then, we construct the Weyl unitaries

$$W_\theta(\mathbf{m}) := \mathrm{e}^{-i\pi\theta m_1 m_2}\ u^{m_1}\, v^{m_2}\ ,$$

indexed by two-dimensional vectors with integral components $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2$. The operators $W_\theta(\mathbf{m})$ satisfy the Weyl relations

$$W_\theta(\mathbf{m})W_\theta(\mathbf{n}) = \mathrm{e}^{i\pi\theta\sigma(\mathbf{m},\mathbf{n})}\ W_\theta(\mathbf{m}+\mathbf{n}) = \mathrm{e}^{2i\pi\theta\sigma(\mathbf{m},\mathbf{n})}\ W_\theta(\mathbf{n})W_\theta(\mathbf{m})\ , \tag{4.1}$$

with σ the symplectic form

$$\sigma(\mathbf{m}, \mathbf{n}) := m_1 n_2 - m_2 n_1\ .$$

Notice that $W_\theta(\mathbf{m})^* = W_\theta(-\mathbf{m})$ and that products $\prod_j W_\theta(\mathbf{m}_j)$ are reducible to just a single $W_\theta(\sum_j \mathbf{m}_j)$ multiplied by a phase.

There exists a unique C*-norm on the linear span of the Weyl unitaries and the C*-algebra $\mathfrak{A}$ is the completion with respect to that norm of $\{W_\theta(f) \mid f$ complex function on $\mathbf{Z}^2$ with bounded support$\}$, where

$$W_\theta(f) := \sum_{\mathbf{m}\in\mathcal{F}} f(\mathbf{m})\, W_\theta(\mathbf{m})\ . \tag{4.2}$$

The dynamics Θ is determined by the linear extension to $\mathfrak{A}$ of the map

$$\Theta(W_\theta(\mathbf{m})) := W_\theta(T\mathbf{m})\ . \tag{4.3}$$

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with integer entries such that $ad - bc = 1$ and $a + d > 2$. In this way, the eigenvalues $\lambda_+ = \lambda > 1$ and $\lambda_- = \lambda^{-1}$ of T are irrational and the associated

eigenvectors $\mathbf{v}_\pm$ correspond to stretching, respectively shrinking, directions with positive, respectively negative, Lyapounov exponent $\pm\log\lambda$. Since $\sigma(T\mathbf{m},T\mathbf{n}) = \sigma(\mathbf{m},\mathbf{n})$, the map in (4.3) preserves the commutation relations (4.1) and thus extends to an automorphism of $\mathfrak{A}$.

Finally, the reference state ϕ is chosen to be the unique tracial state on $\mathfrak{A}$ given by

$$\phi\big(W_\theta(\mathbf{m})\big) := \delta_{\mathbf{0},\mathbf{m}} \ . \tag{4.4}$$

The state ϕ is clearly Θ-invariant.

Remark. Via the GNS construction an infinite-dimensional Hilbert space representation of $\mathfrak{A}$ is obtained, even in the case of rational deformation parameters θ. This makes the quantization procedure introduced above drastically different from the ones presented in the literature [BBTV,D] where θ is rational, typically $1/N$ and periodicity is imposed, namely

$$u^N = e^{i\alpha}\ \mathbb{1}\,, \qquad v^N = e^{i\beta}\ \mathbb{1}\ .$$

In such cases, the resulting algebra acts on an N-dimensional Hilbert space and the quantized hyperbolic automorphisms of the torus are a useful testing ground for studying the classical limit of quantized classically chaotic systems. Indeed, notice that by setting $\theta = 0$ (or letting $N \to \infty$), the Weyl commutation relations (4.1) are implemented by the exponential functions on the torus. This means that the algebra $\mathfrak{A}$ becomes the Abelian C*-algebra of continuous functions on the torus which can be used to construct the corresponding algebraic classical dynamical system.

4.1 Number theoretical interlude

Contrary to the finite-dimensional quantization procedures mentioned above, where quasi-periodicity spoils any true relaxation property, the dynamical systems $(\mathfrak{A}, \Theta, \phi)$ enjoy, depending on θ, sufficiently strong clustering properties. The behaviour of commutators of observables largely separated in time is determined by the value of the deformation parameter

$$\Big[W_\theta(\mathbf{m})\,,\,W_\theta(T^t\mathbf{n})\Big] = \Big(1 - \mathrm{e}^{2\pi i\theta\sigma(\mathbf{m},T^t\mathbf{n})}\Big)W_\theta(\mathbf{m}+T^t\mathbf{n})\ ,\qquad t\in\mathbf{Z}\ .$$

If θ is rational, $\theta\sigma(\mathbf{m},T^t\mathbf{n})$ mod 1 can only assume a finite number of values, and hence no clear limiting behaviour occurs, unless of course $\theta = 0$. Some interesting behaviour occurs for peculiar irrational values of the deformation parameter. We shall first examine such a possibility in more detail and study the convergence mod 1 of the exponent $\theta\sigma(\mathbf{m},T^t\mathbf{n})$ for all $\mathbf{m},\mathbf{n}\in\mathbf{Z}^2$ when $t\to\pm\infty$. We adapt to the present problem a technique used in [AGL] and obtain as a sub-case a result derived in [N] with other means.

Proposition 5. *Let* $T = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ *with* $a,b,c,d\in\mathbf{Z}$ *such that* $ad-bc=1$ *and* $a+d>2$. *For any* $\mathbf{m}=(m_1,m_2)$ *and* $\mathbf{n}=(n_1,n_2)\in\mathbf{Z}^2$*, define*

$$\Delta(\mathbf{m},\mathbf{n}) := (1-a)(m_1n_2+n_1m_2) - c\,m_1n_1 - b\,m_2n_2\ .$$

Let $\lambda>1$ *and* λ^{-1} *be the irrational eigenvalues of* T*,* $\mathbf{v}_\pm$ *the corresponding eigenvectors and set*

$$\beta(t) := \mathrm{Tr}\,T^t = \lambda^t + \lambda^{-t}\ ,\qquad t\in\mathbf{Z}\ . \tag{4.5}$$

Let $\sigma(\mathbf{m},\mathbf{n})$ *be the symplectic form* $m_1n_2 - m_2n_1$ *on* $\mathbf{Z}^2\times\mathbf{Z}^2$ *and let* $\theta\in[0,1)$ *be irrational, then, the limits*

$$q_\pm(\mathbf{m},\mathbf{n}) := \lim_{t\to\pm\infty}\theta\ \sigma\big(\mathbf{m},\ T^t\mathbf{n}\big)\mod 1 \tag{4.6}$$

exist iff the following limit exists

$$\beta := \lim_{t\to+\infty}\frac{\theta}{\lambda^2-1}\,\beta(t)\mod 1\ . \tag{4.7}$$

When the limit in (4.7) exists, its possible values are the rational numbers

$$\beta_r = \frac{r}{\beta(1)-2}\ ,\quad 0\le r\le\beta(1)-3\ . \tag{4.8}$$

In such cases, the limits (4.6) are of the form

$$q^r_\pm(\mathbf{m},\mathbf{n}) = \pm\beta_r\,\Delta(\mathbf{m},\mathbf{n})\mod 1\ .$$

Proof: Let $\mathbf{v}_\pm$ be the normalized eigenvectors of T. They satisfy $T\mathbf{v}_\pm = \lambda^{\pm 1}\mathbf{v}_\pm$ with components with respect to the basis $\{(1,0),\ (0,1)\}$ given by

$$v_\pm(1) = b/\sqrt{b^2 + (a-\lambda^{\pm 1})^2}\ , \quad v_\pm(2) = (\lambda^{\pm 1} - a)/\sqrt{b^2 + (a-\lambda^{\pm 1})^2}\ .$$

By expanding $\mathbf{m} \in \mathbf{Z}^2$ as $\mathbf{m} = c_+(\mathbf{m})\mathbf{v}_+ + c_-(\mathbf{m})\mathbf{v}_-$ and using $(a-\lambda)(a-\lambda^{-1}) = -bc$, we explicitly compute

$$\begin{aligned}\sigma\big(\mathbf{m},\ T^t\mathbf{n}\big) = &\ \frac{m_1 n_2 \lambda^{t+2} + m_2 n_1 \lambda^t - \lambda^{t+1}(c\, m_1 n_1 + b\, m_2 n_2 + a\, m_1 n_2 + a\, m_2 n_1)}{\lambda^2 - 1} \\ &- \frac{m_1 n_2 \lambda^{-t} + m_2 n_1 \lambda^{-t+2} - \lambda^{-t+1}(c\, m_1 n_1 + b\, m_2 n_2 + a\, m_1 n_2 + a\, m_2 n_1)}{\lambda^2 - 1}\ .\end{aligned}$$

Then, setting $\phi := \theta/(\lambda^2 - 1)$, one works out that

$$\begin{aligned} q_+(\mathbf{m},\mathbf{n}) &= \lim_{t\to+\infty} \phi\Big(m_1 n_2 \beta(t+2) + m_2 n_1 \beta(t) - \Delta_1(\mathbf{m},\mathbf{n})\beta(t+1)\Big) \mod 1 \\ q_-(\mathbf{m},\mathbf{n}) &= -\lim_{t\to-\infty} \phi\Big(m_1 n_2 \beta(-t) + m_2 n_1 \beta(-t+2) - \Delta_1(\mathbf{m},\mathbf{n})\beta(-t+1)\Big) \mod 1\ ,\end{aligned}$$

where $\Delta_1(\mathbf{m},\mathbf{n}) := a(m_1 n_2 + n_1 m_2) + c m_1 n_1 + b m_2 n_2$. Therefore, if the limit in (4.7) exists, then the two limits above exist. Vice versa, if the previous two limits exist for all $\mathbf{m},\mathbf{n} \in \mathbf{Z}^2$, then, choosing proper values for $\mathbf{m}$ and $\mathbf{n}$, the limit in (4.7) exists as well.

The traces in (4.5) obey the recursion relations

$$\beta(t) = \beta(1)\beta(t-1)\ -\ \beta(t-2)\ , \quad t \geq 2\ ,$$

with $\beta(0) = 2$ and $\beta(1) = \lambda + \lambda^{-1}$. Thus, if the limit in (4.7) exists, β is determined by

$$\beta\left(\beta(1) - 2\right) = 0 \mod 1\ ,$$

and ranges $\bmod 1$ among the rationals in (4.8). The explicit form of the limit $q_\pm(\mathbf{m},\mathbf{n})$ is obtained by inserting in the expressions for $q_\pm(\mathbf{m},\mathbf{n})$ the values of β_r.

■

The previous proposition gives the explicit form of the limits in (4.7) if they exist. The next result concerns the values of θ such that this is indeed the case.

Proposition 6. *$Lim_t\, \beta(t)\theta/(\lambda^2-1) \mod 1$ exists and equals $\beta_r = r/(\beta(1)-2)$, with $0 \leq r \leq \beta(1) - 3$, iff*

$$\theta = \theta^r_\ell := \lambda\,\ell + (\lambda - 1)\,\beta_r \mod 1\,, \qquad \ell \in \mathbf{Z}\ . \tag{4.9}$$

Proof: We define $\phi := \theta/(\lambda^2 - 1)$ and divide the proof in three steps.
First, we write $\phi\,\beta(t) = a(t) + b(t)$, where $a(t)$ is the largest natural number smaller than $\phi\ \beta(t)$. If the limit in (4.7) exists, then $\lim_t b(t) = \beta_r$ for $0 \le r \le \beta(1) - 3$. Let $\epsilon(t) := b(t) - \beta_r$ and consider the quantities $\psi(t) := \phi\beta(t) - \beta_r = a(t) + \epsilon(t)$. They are easily showed to obey the recursion relations

$$\psi(t+2) = \beta(1)\psi(t+1) - \psi(t)\ +\ r\ , \qquad t \ge 0\ .$$

The latter can be rewritten as

$$a(t+2) - \beta(1)a(t+1) + a(t) - r = \beta(1)\epsilon(t+1) - \epsilon(t+2) - \epsilon(t)\ ,$$

whence we deduce that there must exist an integer T such that for $t \ge T$ both sides of the above equality vanish. Indeed, the l.h.s. is an integer, while the r.h.s. goes to zero with $t \to +\infty$. In particular, this argument yields

$$a(T+t+2) = \beta(1)\ a(T+t+1) - a(T+t) +\ r\ , \qquad t \ge 0\ . \tag{4.10}$$

We prove by induction that for $t \ge 0$,

$$a(T+t+2) := \gamma(t+1)\, a(T+1) - \gamma(t)\, a(T) + r\sum_{k=0}^{t} \gamma(k)\ , \tag{4.11}$$

with $\gamma(t)$ the integers such that $\gamma(0) = 1$, $\gamma(1) = \beta(1)$ and

$$\gamma(t+2) = \beta(1)\,\gamma(t+1) - \gamma(t)\ , \qquad t \ge 0\ . \tag{4.12}$$

The case $t = 0$ is obvious. Suppose that (4.11) holds for $0 \le t \le s-1$ and rewrite (4.10) with $t = s$ as

$$\begin{aligned} a(T+s+2) = a(T+1)\ \big(\beta(1)\gamma(s) - \gamma(s-1)\big) - a(T)\ \big[\beta(1)\gamma(s-1) - \gamma(s-2)\big] \\ + r\ \Big(1 + \beta(1) + \sum_{k=0}^{s-2}\big(\beta(1)\gamma(k+1) - \gamma(k)\big)\Big)\ . \end{aligned}$$

Using (4.12), $a(T+s+2)$ turns out to be of the form (4.11) with $t = s$.

The second step consists in observing that the coefficients

$$c(t) := \frac{\lambda^{t+2} - \lambda^{-t}}{\lambda^2 - 1}$$

of the expansion of $(z^2\ -\ \beta(1)\ z\ +\ 1)^{-1}$ around $z = 0$ fulfil the same recursion relations as the $\gamma(t)$ in (4.12), with the same initial conditions. Therefore, $\gamma(t) = c(t)$ for all $t \ge 0$

and one has

$$\beta(t) = (\lambda^2 - 1)\ \gamma(t-2) + \lambda^{-t}\ \frac{1+\lambda^2}{\lambda^2} \tag{4.13}$$

$$\gamma(t-1) = \gamma(t-2) + 1 + (\beta(1) - 2) \sum_{k=0}^{t-2} \gamma(k) \qquad t \geq 2 \tag{4.14}$$

$$\lambda^{-t} = \gamma(t) - \lambda\ \gamma(t-1)\ , \qquad t \geq 1 \tag{4.15}$$

$$\sum_{k=0}^{t-2} \gamma(k) = -\frac{\lambda}{(\lambda^2-1)} + \frac{\lambda^{t+1} + \lambda^{-t+2}}{(\lambda^2-1)(\lambda-1)}\ . \tag{4.16}$$

We can use the previous relations to prove that the limit in (4.7) exists and equals β_r in (4.8) iff

$$\theta = \lambda^{-T} \Big[\lambda\ a(T+1) - a(T) + \beta_r(\lambda - 1) \Big] \mod 1\ . \tag{4.17}$$

We write $\phi = \theta/(\lambda^2 - 1)$ and $\phi\beta(T+t) = a(T+t) + b(t)$ as in the beginning of this proof. Then, dividing by $\beta(T+t)$ and letting $t \to +\infty$, one recovers (4.17) by means of (4.5), (4.11), (4.13) and (4.16). Vice versa, if θ is as in (4.17), (4.5) and (4.13) yield

$$\begin{aligned} &\lim_{t\to+\infty} \beta(t)\ \frac{\lambda^{-T}}{\lambda^2-1}\ \Big(\lambda\ a(T+1) - a(T)\Big) \mod 1 = 0, \\ &\lim_{t\to+\infty} \beta(t)\ \lambda^{-T}\ \beta_r(\lambda-1) \mod 1 = \lim_{t\to+\infty} \beta_r\big(\beta(t+1) - \beta(t)\big) \mod 1 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = \lim_{t\to+\infty} \beta_r\big(\gamma(t-1) - \gamma(t-2)\big) \mod 1\ , \end{aligned}$$

so sufficiency follows from (4.14), since $\beta_r = r/(\beta(1) - 2)$.

In the third and last step we reduce the expression (4.17) to the simpler form (4.9) by using the fact that, in (4.17), the integers T, $a(T+1)$ and $a(T)$ are so far undetermined. In fact, the sufficiency of (4.9) to guarantee the existence of the limit (4.7) is proved along the same lines as for that of (4.17) above. For the necessity, we have to show that (4.17) reduces to (4.9). Using (4.15), (4.17) reads

$$\theta = \lambda\ d + \beta_r(\lambda-1)\Big(\gamma(T) - \lambda\ \gamma(T-1)\Big) \mod 1\ ,$$

where $d := \gamma(T-1)\ a(T) - \gamma(T-2)\ a(1+T)$. By using (4.14) and the fact that

$$\beta_r = \frac{r}{\beta(1)-2} = r\,\frac{\lambda}{(\lambda-1)^2}\ , \quad \text{for} \quad 0 \leq r \leq \beta(1) - 3\ ,$$

we can rewrite

$$\beta_r\ (\lambda-1)\Big(\gamma(T) - \lambda\ \gamma(T-1)\Big) = \beta_r(\lambda-1) + d_1 + \lambda\,\delta\ ,$$

where $d_1 := d + r\ \sum_{p=0}^{T-2} \gamma(p)$ and $\delta := -r \sum_{p=0}^{T-1} \gamma(p)$. Since d_1 and δ are generic integers, the expression (4.9) is obtained with $\ell = d + d_1$.

■

The following and last number theoretical result is a simple characterization of certain powers of the evolution matrix T.

Proposition 7. *Let T be a 2×2 hyperbolic matrix with integer entries, unit determinant and trace $\beta(1) > 2$ as in Propositions 5 and 6. Then, the matrix $T^{\beta(1)-2}$ is congruent to the identity matrix $\mathrm{mod}\,(\beta(1) - 2)$, in the sense that its diagonal entries are congruent to $1 \mod (\beta(1) - 2)$ and the off-diagonal ones are congruent to $0 \mod (\beta(1) - 2)$.*

Proof: The eigenvalues λ and λ^{-1} of T solve $z^2 - \beta(1)z + 1 = 0$, therefore they satisfy the matricial recursion relations

$$\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix} = \beta(1) \begin{pmatrix} \lambda^{n-1} & 0 \\ 0 & \lambda^{-(n-1)} \end{pmatrix} - \begin{pmatrix} \lambda^{n-2} & 0 \\ 0 & \lambda^{-(n-2)} \end{pmatrix} \quad \text{for} \quad n \geq 3 \ .$$

Because the eigenvalues of T are non-degenerate we can find a similarity transformation which allows us to diagonalize T, such that the above equality may be turned into

$$T^n = \beta(1)\, T^{n-1} - T^{n-2} \ .$$

Then, defining $S_n := T^n - \mathbb{1}$

$$S_n = \beta(1)\, S_{n-1} - S_{n-2} + \beta(1) - 2 \ , \quad \text{with} \quad S_0 = 0 \quad \text{and} \quad S_1 = T - \mathbb{1} \ ,$$

the differences $S_k - S_{k-1}$, $k \geq 2$, are connected by the following relation

$$S_k - S_{k-1} = S_{k-1} - S_{k-2} + \big(\beta(1) - 2\big)\big(S_{k-1} + 1\big) \ ,$$

whence, by telescopic summation,

$$S_n = \big(\beta(1) - 2\big)\big(\sum_{k=2}^{n} S_{k-1} + n - 1\big) + S_{n-1} + S_1 \ .$$

Since all matrices S_n have integral entries, $S_n \equiv S_1 + S_{n-1} \mod (\beta(1) - 2)$ and, by iterating n times the previous congruence,

$$S_n \equiv n\, S_1 \mod (\beta(1) - 2) \ .$$

Thus, when $n = \beta(1) - 2$, S_n turns out to be congruent to $0 \mod (\beta(1) - 2)$ and the result follows.

■

4.2 Clustering properties and statistics of fluctuations

We return now to the discussion of the ergodic properties of the quantized hyperbolic automorphisms of the torus $(\mathfrak{A}, \Theta, \phi)$ and of their dependence on the deformation parameter $\theta \in [0, 1)$.

Proposition 8. *For all $\theta \in [0,1)$, the quantized hyperbolic automorphisms of the torus $(\mathfrak{A}, \Theta, \phi)$ are weakly clustering in the sense of (3.1). They are strongly clustering in the sense of (3.2) iff, according to Proposition 5,*

$$\lim_{t\to+\infty} \frac{\theta}{\lambda^2 - 1}\beta(t) = 0 \mod 1 \ ,$$

that is, according to Proposition 6, iff $\theta = \lambda\ell \mod 1$, with $\ell \in \mathbf{Z}$.

Proof: The state ϕ is tracial and the operators $W_\theta(f)$ in (4.2), with finitely supported f on $\mathbf{Z}^2$, are uniformly dense in $\mathfrak{A}$. Therefore, by linearity, the cluster property in (3.1) is proved by showing that

$$\begin{aligned} &\lim_{t\to\infty} \phi\Big(W_\theta(\mathbf{m})W_\theta(T^t\mathbf{n})W_\theta(\mathbf{p})\Big) = \\ &\qquad \lim_{t\to\infty} \mathrm{e}^{\pi i\theta\left(\sigma(\mathbf{m},T^t\mathbf{n})+\sigma(\mathbf{m}+T^t\mathbf{n},\mathbf{p})\right)}\,\delta_{\mathbf{0},\mathbf{m}+T^t\mathbf{n}+\mathbf{p}} = 0 \ , \end{aligned}$$

for generic integer vectors $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{p} \in \mathbf{Z}^2$. The commutation relations (4.1) and the definition (4.4) of the state ϕ have been used to derive the equality of above. The limit is equal to zero because of the hyperbolic character of the matrix T. Indeed, by means of the eigenvectors $\mathbf{v}_\pm$, one puts into evidence the Lyapounov exponent $\log\lambda$,

$$T^t\mathbf{n} = c_+(\mathbf{n})\lambda^t\mathbf{v}_+ + c_-(\mathbf{n})\lambda^{-t}\mathbf{v}_- \ .$$

It is thus evident that in the limit of large $|t|$, the condition $\mathbf{m} + T^t\mathbf{n} + \mathbf{p} = 0$ cannot be fulfilled.

Analogously, one calculates

$$\phi\Big(\Big[W_\theta(\mathbf{m})\,,\,W_\theta(T^t\mathbf{n})\Big]^*\Big[W_\theta(\mathbf{m})\,,\,W_\theta(T^t\mathbf{n})\Big]\Big) = 4\sin^2\big(\pi\theta\sigma(\mathbf{m},T^t\mathbf{n})\big) \ .$$

Strong clustering (3.2) implies that the right hand side of the above equation must vanish, that is that $\lim_t \theta\,\sigma(\mathbf{m},T^t\mathbf{n}) \mod 1 = 0$, for all $\mathbf{m}$ and $\mathbf{n} \in \mathbf{Z}^2$. On the other hand, if $\lim_t \theta\sigma(\mathbf{m},T^t\mathbf{n}) \mod 1$ equals 0 for all integer vectors $\mathbf{m}$ and $\mathbf{n}$, then all commutators as the one above vanish in norm and therefore the dynamical system is even norm-asymptotic Abelian and hence, as it is weakly clustering, it is also strongly clustering.

■

Next we shall compute the asymptotic state ϕ_∞ and the fluctuations for the hyperbolic toral automorphisms $(\mathfrak{A}, \Theta, \phi)$ with deformation parameter θ. We shall see that, depending on the choice of the deformation parameter, these aspects can be quite different. We shall rely on formula (2.4) with single-time averages as in (2.3). The "words" in the asymptotic free algebra consist now of free products of elements of the type $W_\theta(f^{(j)})_{\nu(j)}$ with $W_\theta(f)$ given by (4.2). Because of linearity of $f \mapsto W_\theta(f)$ we may restrict ourselves to studying multi-time correlation functions of Weyl operators

$$\mathbf{t} \mapsto \phi\Big(W_\theta\big(T^{t_{\nu(1)}}\mathbf{n}^{(1)}\big)\, W_\theta\big(T^{t_{\nu(2)}}\mathbf{n}^{(2)}\big) \cdots W_\theta\big(T^{t_{\nu(n)}}\mathbf{n}^{(n)}\big)\Big) \ ,$$

where ν maps $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots, s\}$.

The following lemma can be found in [NT2]. For the sake of completeness we provide the proof as well.

Lemma 1. *Let T be any hyperbolic matrix defining the dynamics of $(\mathfrak{A}, \Theta, \phi)$ as in (4.3) and let σ be the symplectic form $\sigma(\mathbf{m}, \mathbf{n}) = m_1 n_2 - m_2 n_1$ with $\mathbf{m}$, $\mathbf{n} \in \mathbf{Z}^2$. For almost all $\theta \in [0, 1)$, the orbits $k \in \mathbf{Z} \mapsto \theta\sigma(\mathbf{m}, T^k\mathbf{n})$ are uniformly distributed over the circle $\mathbf{T}$ of perimeter 1.*

Proof: Uniform distribution means that $\forall f \in C(\mathbf{T})$ we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\theta\sigma(\mathbf{m}, T^k\mathbf{n})) = \int_0^1 dx\, f(x) \ .$$

This is equivalent to requiring that

$$\overline{f}_N(\theta) := \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{2\pi i n\theta\sigma(\mathbf{m}, T^k\mathbf{n})} \longrightarrow 0 \qquad \forall n \in \mathbf{Z}_0 \ .$$

To see whether this holds for almost all θ, we consider

$$\int_0^1 d\theta\, |\overline{f}_N(\theta)|^2 = \frac{1}{N^2} \sum_{k,k'=1}^{N} \int_0^1 d\theta\, \mathrm{e}^{2\pi i n\theta\sigma(\mathbf{m}, (T^k - T^{k'})\mathbf{n})} \ .$$

Because $\sigma\big(\mathbf{m}, (T^k - T^{k'})\mathbf{n}\big)$ is an integer, we have that

$$\int_0^1 d\theta\, \mathrm{e}^{2\pi i n\theta\sigma(\mathbf{m}, (T^k - T^{k'})\mathbf{n})} = \begin{cases} 1 & \text{if } \sigma\big(\mathbf{m}, (T^k - T^{k'})\mathbf{n}\big) = 0 \\ 0 & \text{else} \end{cases} \ .$$

And, because $\sigma\big(\mathbf{m}, (T^k - T^{k'})\mathbf{n}\big) = 0 \ \forall\ \mathbf{m} \in \mathbf{Z}^2 \Leftrightarrow T^k\mathbf{n} = T^{k'}\mathbf{n} \Leftrightarrow k = k'$, we have that

$$\int_0^1 d\theta\, |\overline{f}_N(\theta)|^2 = 1/N^2 \sum_{k=k'=1}^{N} 1 = 1/N \longrightarrow 0 \text{ for } N \to \infty \ . \qquad \blacksquare$$

Using the previous lemma and Propositions 5 and 6, we deduce

Corollary 4. *a) There exists a set $\mathcal{Z} \subset [0,1)$ of measure 1 such that for all $\theta \in \mathcal{Z}$ and for all $\mathbf{m}$, $\mathbf{n} \in \mathbf{Z}^2$ $(\neq \mathbf{0})$, we have*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathrm{e}^{2\pi i \theta \sigma(\mathbf{m}, T^t \mathbf{n})} = 0 \ .$$

b) If $\lim_t \theta\sigma(\mathbf{m}, T^t\mathbf{n}) \mod 1$ exists, then

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathrm{e}^{2\pi i \theta \sigma(\mathbf{m}, T^t \mathbf{n})} = \mathrm{e}^{2\pi i \beta_r \Delta(\mathbf{m},\mathbf{n})} \ ,$$

where β_r and $\Delta(\mathbf{m}, \mathbf{n})$ are defined in Proposition 5.

In order to proceed with the construction of an asymptotic state on $\mathfrak{A}_\infty$ following the prescription of (2.4), we need a preliminary technical result which is proved in [ABDF].

Lemma 2. *For $d, k \in \mathbf{N}$ define*

$$\Delta_d^k(t_1, \ldots, t_k) = \begin{cases} 0 & \textit{if } |t_i - t_j| \leq d \textit{ for some } 1 \leq i \neq j \leq k \\ 1 & \textit{else} \end{cases} \ .$$

Then, if the multiple average (2.14) of a uniformly bounded function $f : \mathbf{N}^k \to \mathbf{C}$ exists, we have

$$\begin{aligned} &\mathrm{Avg}\Big(t_k \mapsto \cdots \mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto f(t_1, t_2, \ldots, t_k)\Big)\Big)\cdots\Big) = \\ &\mathrm{Avg}\Big(t_k \mapsto \cdots \mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto \Delta_d^k(t_1, \ldots, t_k) f(t_1, t_2, \ldots, t_k)\Big)\Big)\cdots\Big) \ . \end{aligned}$$

Proposition 9 and the subsequent corollary deal with the asymptotic structure emerging from the first possibility of corollary 4, while proposition 10 and its corollary deal with the second case.

Proposition 9. *Let us consider a hyperbolic toral automorphism determined by a deformation parameter $\theta \in \mathcal{Z}$ as in corollary 4, part a). The multiple average in (2.4), with single-time averages as in (2.3), exists and the state ϕ_∞ it defines on $\mathfrak{A}_\infty$ is permutation invariant as in (2.5).*

Proof: Because of linearity, we need only to consider expectations of words of the type

$$\phi_\infty\Big(W_\theta\big(\mathbf{n}^{(1)}\big)_{\nu(1)} W_\theta\big(\mathbf{n}^{(2)}\big)_{\nu(2)} \cdots W_\theta\big(\mathbf{n}^{(n)}\big)_{\nu(n)}\Big) \ ,$$

which are to be computed as the multiple averages

$$\mathrm{Avg}\Big(t_s \mapsto \cdots \mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto \\ \phi\Big(W_\theta\big(T^{t_{\nu(1)}}\mathbf{n}^{(1)}\big)\, W_\theta\big(T^{t_{\nu(2)}}\mathbf{n}^{(2)}\big) \cdots W_\theta\big(T^{t_{\nu(n)}}\mathbf{n}^{(n)}\big)\Big)\Big)\Big) \cdots\Big) \;.$$

ν maps $\{1,2,\ldots,n\}$ into $\{1,2,\ldots,s\}$ and we may assume that all Weyl operators are different from the identity, that is that none of the $\mathbf{n}^{(j)} = 0$.
Before taking the average we group all Weyl operators belonging to a same time, gathering hereby a phase factor according to the commutation relations (4.1). Then, using Lemma 2, we reduce the problem to the computation of the multiple time-average of the phase factors. By the assumption on the deformation parameter θ, these averages are always zero unless they are time-independent. This proves the permutation invariance of ϕ_∞.

With $p \in \{1,2,\ldots,s\}$, let I_p denote the set of natural numbers $i \in \{1,2,\ldots,n\}$ such that $\nu(i) = p$. For any given $p \in \{1,2,\ldots,s\}$, the Weyl operators $W_\theta\big(T^{t_{\nu(i)}}\mathbf{n}^{(i)}\big)$, with $i \in I_p$, can be brought together, from right to left, to form the composite word

$$W_{I_p}(t_p) := W_\theta\big(T^{t_p}\mathbf{n}^{(i_1)}\big)\, W_\theta\big(T^{t_p}\mathbf{n}^{(i_2)}\big) \cdots W_\theta\big(T^{t_p}\mathbf{n}^{(i_{c_p})}\big) \;.$$

Using (4.1), the regrouping produces a phase factor

$$\exp\Big(\sum_{i\in I_p}\sum_{k\in K(i)}\sum_{j\in J_i(k)} 2\pi i\theta\sigma\Big(T^{t_k}\mathbf{n}^{(j)},\, T^{t_p}\mathbf{n}^{(i)}\Big)\Big) \;.$$

The index set $K(i)$ specifies those times $t_k \neq t_p$ that are encountered while commuting $W_\theta\big(T^{t_p}\mathbf{n}^{(i)}\big)$, $i \in I_p$, over the various $W_\theta\big(T^{t_{\nu(i)}}\mathbf{n}^{(i)}\big)$ in order to concatenate the former word to the right of a previous word $W_\theta\big(T^{t_p}\mathbf{n}^{(j)}\big)$, $j \in I_p$.

We shall regroup the Weyl operators by first bringing together all those at time t_1, then all those at time t_2 and so on. We end up with

$$\phi\Big(W_\theta\big(T^{t_{\nu(1)}}\mathbf{n}^{(1)}\big)\, W_\theta\big(T^{t_{\nu(2)}}\mathbf{n}^{(2)}\big) \cdots W_\theta\big(T^{t_{\nu(n)}}\mathbf{n}^{(n)}\big)\Big) = \\ F\Big(t_1,t_2,\ldots,t_s\Big)\, \phi\Big(W_{I_1}(t_1)\, W_{I_2}(t_2)\, \cdots W_{I_s}(t_s)\Big) \;, \tag{4.18}$$

where

$$\begin{aligned} F\Big(t_1,t_2,\ldots,t_s\Big) &= \prod_{p=1}^{s} F_p(t_p,t_{p+1},\ldots,t_s) \qquad \text{with} \\ F_p(t_p,t_{p+1},\ldots,t_s) &= \exp\Big(\sum_{i\in I_p}\sum_{k\in K_{p+1}(i)}\sum_{j\in J_i(k)} 2\pi i\theta\sigma\Big(T^{t_k}\mathbf{n}^{(j)},\, T^{t_p}\mathbf{n}^{(i)}\Big)\Big) \;, \end{aligned} \tag{4.19}$$

and $K_{p+1}(i)$ labels times t_k with $k \geq p+1$. Next, using Lemma 2, we can always restrict ourselves in the multiple time-averages to times $t_{\nu(j)}$ that are sufficiently separated from one another. As the dynamics $\mathbf{n} \mapsto T\mathbf{n}$ is hyperbolic on $\mathbf{Z}^2$, the separation between different times can always be chosen in such a way that

$$\begin{aligned} &\nu(i) \neq \nu(j) \Longrightarrow T^{t_{\nu(i)}}\mathbf{n}^{(i)} + T^{t_{\nu(j)}}\mathbf{n}^{(j)} \neq \mathbf{0} \qquad \text{and} \\ &\sum_{j=1}^{n} T^{t_{\nu(j)}}\mathbf{n}^{(j)} = \mathbf{0} \Longleftrightarrow \mathbf{n}_p := \sum_{i\in I_p} \mathbf{n}^{(i)} = \mathbf{0} \qquad \forall p = 1,2,\ldots,s\ . \end{aligned} \tag{4.20}$$

From (4.20) and (4.4), we see that the only possibly non-vanishing averages are those for which $\mathbf{n}_p = \mathbf{0}$ for each p separately, in which case $W_{I_p} = G_p\, W_\theta(\mathbf{n}_p) = G_p\, \mathbb{1}$, G_p being a suitable phase obtained through (4.1). We can therefore write

$$\begin{aligned} &\phi_\infty\Big(W_\theta\big(\mathbf{n}^{(1)}\big)_{\nu(1)}\, W_\theta\big(\mathbf{n}^{(2)}\big)_{\nu(2)} \cdots W_\theta\big(\mathbf{n}^{(n)}\big)_{\nu(n)}\Big) = \\ &\Big(\prod_{q=1}^{s} G_q \delta_{\mathbf{0},\mathbf{n}_q}\Big)\, \mathrm{Avg}\Big(t_s \mapsto \cdots \mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto F\Big(t_1,t_2,\ldots,t_s\Big)\Big)\Big)\cdots\Big)\ . \end{aligned} \tag{4.21}$$

We now take the average with respect to t_1. In the product (3.19), only the factor F_1 can depend on t_1. Since θ belongs to $\mathcal{Z}$, this average is either zero or one. Moreover, it can only be one if F_1 does not depend on t_1. We can now successively average over the consecutive times $t_2, t_3, \ldots$ and conclude that either the average is zero or that it is independent of all $t_1, t_2, \ldots, t_s$. In both cases, ϕ_∞ is permutation invariant in the sense of (2.5).

■

Corollary 5. *A hyperbolic toral automorphism determined by a deformation parameter $\theta \in \mathcal{Z}$ as in Corollary 4, part a), satisfies condition (3.8) and (3.12). Therefore, the fluctuations of a centred self-adjoint element $W_\theta(f)$ in (4.2), are semicircularly distributed.*

Proof: Condition (3.8) is satisfied because of the hyperbolic character of the matrix T which implements the dynamics. Thus, because of linearity and since the reference state is the trace, it suffices to check condition (3.12) in the form

$$\mathrm{Avg}\Big(t \mapsto \phi\Big(W_\theta(\mathbf{p})W_\theta(T^t\mathbf{m})W_\theta(\mathbf{n})W_\theta(-T^t\mathbf{m})\Big)\Big) = 0\ , \tag{4.22}$$

for all $\mathbf{m}$, $\mathbf{n}$, $\mathbf{p} \in \mathbf{Z}^2$. In fact,
a) products of Weyl operators can be reduced to a single Weyl operator multiplied by a phase factor;
b) Weyl operators on the right of $W_\theta(-T^t\mathbf{m})$ can be moved to the left of $W_\theta(T^t\mathbf{m})$ by the tracial properties of the state ϕ and
c) the two and only two Weyl operators at time t carry opposite integral vectors because, otherwise, the hyperbolic character of the matrix T and (4.4) would force the expectation

ϕ, and thus also its average, to vanish asymptotically.
Then (4.22) is obviously true, as

$$\mathrm{Avg}\Big(t \mapsto \phi\Big(W_\theta(\mathbf{p})W_\theta(T^t\mathbf{m})W_\theta(-\mathbf{n})W_\theta(-T^t\mathbf{m})\Big)\Big) = \phi\Big(W_\theta(\mathbf{p})W_\theta(-\mathbf{n})\Big)\ \mathrm{Avg}\Big(t \mapsto \exp\Big(2\pi i\theta\sigma(T^t\mathbf{m},\mathbf{n})\Big)\Big)\ .$$

On the basis of this result, we want to compute $\lim_N \phi_\infty\Big(F_N(W_\theta(f))^{2n}\Big)$ where F_N is the local fluctuation defined in (3.7). Notice that $W_\theta(f)$ is self-adjoint iff, for any $\mathbf{n} \in \mathbf{Z}^2$, $\overline{f(\mathbf{n})} = f(-\mathbf{n})$ and it is centred iff $f(\mathbf{0}) = 0$. We shall denote by $\mathcal{F}$ the support of f. Since ϕ_∞ is permutation invariant, we use Corollary 3 and obtain a semicircular distribution for $W_\theta(f)$ with variance

$$\sigma = \phi(W_\theta(f)^2) = \|f\|_2^2 = \sum_{\mathbf{q}\in\mathcal{F}} |f(\mathbf{q})|^2\ .$$

$\blacksquare$

Remark The previous proof could suggest that the state ϕ_∞ which was constructed in Proposition 9 is a free product of traces, namely that

$$\phi_\infty\left(X^{(1)}_{j_1}X^{(2)}_{j_2}\cdots X^{(n)}_{j_n}\right) = 0$$

whenever $\phi\Big(X^{(k)}_{j_k}\Big) = 0$ and $j_k \neq j_{k+1}$ for all k. This is, however, not the case. An easy counterexample is obtained by considering a correlation function that is independent of a time appearing in the product of observables such as

$$t \mapsto \phi\Big(W_\theta(\mathbf{n})W_\theta(T^t\mathbf{n})W_\theta(-2\mathbf{n})W_\theta(T^t\mathbf{n})W_\theta(\mathbf{n})W_\theta(-2T^t\mathbf{n})\Big) \qquad \mathbf{n} \neq \mathbf{0}\ .$$

Each of the observables in the correlation function is centred but, instead of vanishing,

$$\phi_\infty\Big(W_\theta(\mathbf{n})_1W_\theta(\mathbf{n})_2W_\theta(-2\mathbf{n})_1W_\theta(\mathbf{n})_2W_\theta(\mathbf{n})_1W_\theta(-2\mathbf{n})_2\Big) = 1\ .$$

We shall now see that, when the deformation parameter θ is chosen not to belong to the dense set $\mathcal{Z}$, but, instead, equals any of the special values for which the second part of Corollary 4 holds, then the multiple time-average (2.4) still defines an asymptotic state ϕ_∞, but, except when $\beta_r = 0$, it is not permutation invariant.

Proposition 10. *If the conditions of Corollary 4, part b), are fulfilled, then the multiple average (2.4) exists and it defines an asymptotic state ϕ_∞ on the asymptotic free algebra $\mathfrak{A}_\infty$.*

Proof: We can follow the proof of Proposition 9. The asymptotic state ϕ_∞ is determined by the expectations

$$\phi_\infty\Big(W_\theta\big(\mathbf{n}^{(1)}\big)_{\nu(1)}\, W_\theta\big(\mathbf{n}^{(2)}\big)_{\nu(2)} \cdots W_\theta\big(\mathbf{n}^{(n)}\big)_{\nu(n)}\Big)\ ,$$

where ν maps $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots, s\}$, $s \le n$, and the vectors $\mathbf{n}^{(j)} \in \mathbf{Z}^2$ may be supposed $\neq \mathbf{0}$. By suitable regrouping as in formula (4.18), the expectations above are well-defined if the multiple time-average (4.21) exists. Thus, we start considering the time t_1. By virtue of the assumption on the existence of the limits of the exponents, we get

$$\begin{aligned}
&\mathrm{Avg}\Big(t_1 \mapsto F\Big(t_1, t_2, \ldots, t_s\Big)\Big) = \\
&\prod_{p=2}^{s} F_p\Big(t_p, \ldots, t_s\Big)\, \mathrm{Avg}\Big(t_1 \mapsto \exp\Big(\sum_{i \in I_1} \sum_{k \in K_2(i)} \sum_{j \in J_i(k)} 2\pi i \theta \sigma\Big(T^{t_k}\mathbf{n}^{(j)},\, T^{t_1}\mathbf{n}^{(i)}\Big)\Big)\Big) = \\
&\prod_{p=2}^{s} F_p\Big(t_p, \ldots, t_s\Big)\ \exp\Big(\sum_{i \in I_1} \sum_{k \in K_2(i)} \sum_{j \in J_i(k)} 2\pi i \beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ . \qquad (4.23)
\end{aligned}$$

In the last equality, we have used the quantities β_r and $\Delta(\mathbf{m}, \mathbf{n})$ which have been introduced in Propositions 5 and 6. Furthermore, the index sets $K_\ell(i)$, $i \in I_q$ and $\ell \ge q + 1$ contain $k \ge \ell$.

When averaging with respect to t_2, the factor $F_2(t_2, t_3, \ldots, t_s)$ tends asymptotically to a term similar to the exponential in formula (4.23) of above, without t_2 dependence. Therefore, we only have to compute the average with respect to t_2 of a contribution of the form

$$\exp\Big(\sum_{i \in I_1} \sum_{j \in J_i(2)} 2\pi i \beta_r \Delta\Big(T^{t_2}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)$$

which may come from the presence of the time t_2 among those indexed by $k \in K_2(i)$ in the exponent in (4.23).

We rewrite the quantity $\Delta(\mathbf{m}, \mathbf{n})$ introduced in Proposition 5 as the scalar product $\Delta(\mathbf{m}, \mathbf{n}) = \langle \mathbf{m}, S\mathbf{n}\rangle$ where $S = \begin{pmatrix} -c & 1-a \\ 1-a & -b \end{pmatrix}$. Since powers of T transform $\mathbf{Z}^2$ into $\mathbf{Z}^2$ and S has integral entries, we use Proposition 7 to deduce that

$$\Delta\Big(T^{m(\beta(1)-2)+s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big) = \langle T^s\mathbf{n}^{(j)},\, S\mathbf{n}^{(i)}\rangle\ + (\beta(1) - 2)\, N(\mathbf{n}^{(i)}, \mathbf{n}^{(j)})\ ,$$

for all $m \in \mathbf{Z}$, with $0 \le s \le \beta(1) - 3$ and $N(\mathbf{n}^{(i)}, \mathbf{n}^{(j)})$ a suitable integer. Moreover, $\beta_r = r/(\beta(1) - 2)$ with $0 \le r \le \beta(1) - 3$, thus the following cyclic properties hold

$$\begin{aligned}
&\exp\Big(\sum_{i \in I_1} \sum_{j \in J_i(2)} 2\pi i \beta_r \Delta\Big(T^{m(\beta(1)-2)+s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big) = \\
&\exp\Big(\sum_{i \in I_1} \sum_{j \in J_i(2)} 2\pi i \beta_r \Delta\Big(T^{s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ . \qquad (4.24)
\end{aligned}$$

After the first two averages, we remain with

$$\mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto F\Big(t_1, t_2, \ldots, t_s\Big)\Big)\Big) =$$

$$\prod_{p=3}^{s} F_p\Big(t_p, \ldots, t_s\Big)\ \exp\Big(\sum_{i\in I_2}\sum_{k\in K_3(i)}\sum_{j\in J_i(k)} 2\pi i\beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big) \times$$

$$\exp\Big(\sum_{i\in I_1}\sum_{k\in K_3(i)}\sum_{j\in J_i(k)} 2\pi i\beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ D_1^2$$

where, using (4.24),

$$D_1^2 := \mathrm{Avg}\Big(t_2 \mapsto \exp\Big(\sum_{i\in I_1}\sum_{j\in J_i(2)} 2\pi i\beta_r \Delta\Big(T^{t_2}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\Big)$$

$$= \frac{1}{\beta(1)-2}\sum_{s=0}^{\beta(1)-3} \exp\Big(\sum_{i\in I_1}\sum_{j\in J_i(2)} 2\pi i\beta_r \Delta\Big(T^{s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ .$$

If we go on and consider the average with respect to t_3, the result is

$$\mathrm{Avg}\Big(t_3 \mapsto \mathrm{Avg}\Big(t_2 \mapsto \mathrm{Avg}\Big(t_1 \mapsto F\Big(t_1, t_2, \ldots, t_s\Big)\Big)\Big)\Big) =$$

$$\prod_{p=4}^{s} F_p\Big(t_p, \ldots, t_s\Big)\ \exp\Big(\sum_{i\in I_3}\sum_{k\in K_4(i)}\sum_{j\in J_i(k)} 2\pi i\beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big) \times$$

$$\exp\Big(\sum_{i\in I_2}\sum_{k\in K_4(i)}\sum_{j\in J_i(k)} 2\pi i\beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big) \times$$

$$\exp\Big(\sum_{i\in I_1}\sum_{k\in K_4(i)}\sum_{j\in J_i(k)} 2\pi i\beta_r \Delta\Big(T^{t_k}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ D_{12}^3\, D_1^2\ ,$$

where

$$D_{12}^3 := \frac{1}{\beta(1)-2}\sum_{s=0}^{\beta(1)-3} \exp\Big(2\pi i\beta_r\Big\{\sum_{i\in I_1}\sum_{j\in J_i(3)} \Delta\Big(T^{s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big) + \sum_{k\in I_2}\sum_{q\in J_k(3)} \Delta\Big(T^{s}\mathbf{n}^{(q)},\, \mathbf{n}^{(k)}\Big)\Big\}\Big)\ .$$

After averaging with respect to the remaining times $t_3, t_4, \ldots, t_s$, we obtain a well-defined asymptotic state ϕ_∞ given by

$$\phi_\infty\Big(W_\theta\big(\mathbf{n}^{(1)}\big)_{\nu(1)}\, W_\theta\big(\mathbf{n}^{(2)}\big)_{\nu(2)} \cdots W_\theta\big(\mathbf{n}^{(n)}\big)_{\nu(n)}\Big) = \prod_{p=2}^{s} D_{12\ldots p-1}^{p}$$

where

$$D_{12\ldots p-1}^{p} := \frac{1}{\beta(1)-2}\sum_{s=0}^{\beta(1)-3} \exp\Big(2\pi i\beta_r \sum_{\ell=1}^{p-1}\sum_{i\in I_\ell}\sum_{j\in J_i(p)} \Delta\Big(T^{s}\mathbf{n}^{(j)},\, \mathbf{n}^{(i)}\Big)\Big)\ . \qquad \blacksquare$$

The asymptotic states just constructed are not permutation invariant as in (2.5). Indeed, when $\beta_r \neq 0$,

$$\phi_\infty\Big(W_\theta(\mathbf{p})_3\, W_\theta(\mathbf{p})_1\, W_\theta(\mathbf{m})_3\, W_\theta(-\mathbf{m})_2\, W_\theta(-\mathbf{p})_1\, W_\theta(\mathbf{m})_2\, W_\theta(-\mathbf{p}-\mathbf{m})_3\Big) \neq$$
$$\phi_\infty\Big(W_\theta(\mathbf{p})_1\, W_\theta(\mathbf{p})_3\, W_\theta(\mathbf{m})_1\, W_\theta(-\mathbf{m})_2\, W_\theta(-\mathbf{p})_3\, W_\theta(\mathbf{m})_2\, W_\theta(-\mathbf{m}-\mathbf{p})_1\Big)\ .$$

By relabelling the summation indices, as in the proof of Proposition 9, the inequality of above comes about if we show that, with shortened notation for the successive averages,

$$\begin{aligned}
&\mathrm{Avg}_3\,\mathrm{Avg}_2\,\mathrm{Avg}_1\Big(\phi\Big(W_\theta\big(T^{t_3}\mathbf{p}\big)\, W_\theta\big(T^{t_1}\mathbf{p}\big)\, W_\theta\big(T^{t_3}\mathbf{m}\big)\, W_\theta\big(-T^{t_2}\mathbf{m}\big)\, W_\theta\big(-T^{t_1}\mathbf{p}\big)\\
&\qquad\qquad W_\theta\big(T^{t_2}\mathbf{m}\big)\, W_\theta\big(-T^{t_3}(\mathbf{m}+\mathbf{p})\big)\Big)\Big) \neq\\
&\mathrm{Avg}_1\,\mathrm{Avg}_2\,\mathrm{Avg}_3\Big(\phi\Big(W_\theta\big(T^{t_3}\mathbf{p}\big)\, W_\theta\big(T^{t_1}\mathbf{p}\big)\, W_\theta\big(T^{t_3}\mathbf{m}\big)\\
&\qquad\qquad W_\theta\big(-T^{t_2}\mathbf{m}\big)\, W_\theta\big(-T^{t_1}\mathbf{p}\big)\, W_\theta\big(T^{t_2}\mathbf{m}\big)\, W_\theta\big(-T^{t_3}(\mathbf{m}+\mathbf{p})\big)\Big)\Big)\ .
\end{aligned}$$

We now bring together from right to left the words belonging to t_1, t_2 and t_3, thus obtaining

$$\begin{aligned}
&W_\theta\big(T^{t_3}\mathbf{p}\big)W_\theta\big(T^{t_1}\mathbf{p}\big)W_\theta\big(T^{t_3}\mathbf{m}\big)W_\theta\big(-T^{t_2}\mathbf{m}\big)W_\theta\big(-T^{t_1}\mathbf{p}\big)W_\theta\big(T^{t_2}\mathbf{m}\big)W_\theta\big(-T^{t_3}(\mathbf{m}+\mathbf{p})\big) =\\
&\quad\exp\Big(2\pi i\theta\Big(\sigma\big(T^{t_2}\mathbf{m},T^{t_1}\mathbf{p}\big)-\sigma\big(T^{t_3}\mathbf{m},T^{t_1}\mathbf{p}\big)\Big)\Big)\,\mathbb{1}\ .
\end{aligned}$$

Then, arguing as in the proof of Proposition 10, we compute

$$\begin{aligned}
&\mathrm{Avg}_3\,\mathrm{Avg}_2\,\mathrm{Avg}_1\Big(\exp\Big(2\pi i\theta\Big(\sigma\big(T^{t_2}\mathbf{m},T^{t_1}\mathbf{p}\big)-\sigma\big(T^{t_3}\mathbf{m},T^{t_1}\mathbf{p}\big)\Big)\Big)\Big) =\\
&\mathrm{Avg}_3\,\mathrm{Avg}_2\Big(\exp\Big(2\pi i\beta_r\Big(\Delta\big(T^{t_2}\mathbf{m},\mathbf{p}\big)-\Delta\big(T^{t_3}\mathbf{m},\mathbf{p}\big)\Big)\Big)\Big) =\\
&\left(\frac{1}{\beta(1)-2}\right)^2\left|\sum_{s=0}^{\beta(1)-3}\exp\Big(2\pi i\beta_r\Delta\big(T^s\mathbf{m},\mathbf{p}\big)\Big)\right|^2\ .
\end{aligned}\qquad(4.25)$$

On the other hand,

$$\begin{aligned}
&\mathrm{Avg}_1\,\mathrm{Avg}_2\,\mathrm{Avg}_3\Big(\exp\Big(2\pi i\theta\Big(\sigma\big(T^{t_2}\mathbf{m},T^{t_1}\mathbf{p}\big)-\sigma\big(T^{t_3}\mathbf{m},T^{t_1}\mathbf{p}\big)\Big)\Big)\Big) =\\
&\mathrm{Avg}_1\,\mathrm{Avg}_2\Big(\exp\Big(2\pi i\theta\sigma\big(T^{t_2}\mathbf{m},T^{t_1}\mathbf{p}\big)\Big)\exp\Big(2\pi i\beta_r\Big(\Delta\big(T^{t_1}\mathbf{p},\mathbf{m}\big)\Big)\Big)\Big) =\\
&\mathrm{Avg}_1\Big(\exp\Big(-2\pi i\beta_r\Delta\big(T^{t_1}\mathbf{p},\mathbf{m}\big)\Big)\exp\Big(2\pi i\beta_r\Delta\big(T^{t_1}\mathbf{p},\mathbf{m}\big)\Big)\Big) = 1\ .
\end{aligned}$$

Clearly, the expression (4.25) can be made different from 1 by suitably choosing $\mathbf{m}$ and $\mathbf{p}$.

Notice that, if $\beta_r = 0$, which is the case when $r = 0$, then (4.25) equals 1 as well. In fact, Proposition 8 tells us that the system $(\mathfrak{A}, \Theta, \phi)$ is strongly asymptotically Abelian and Proposition 2 ensures that the asymptotic state is then automatically permutation invariant.

In order to characterize the statistics of fluctuations in the asymptotic algebra $\mathfrak{A}_\infty$, when the asymptotic state ϕ_∞ is defined as in Proposition 10, we show that, besides being weakly clustering, the dynamical systems $(\mathfrak{A}, \Theta, \phi)$ satisfy condition (3.8).

Proposition 11. *Let $(\mathfrak{A}, \Theta, \phi)$ be a quantized toral automorphism with deformation parameter θ such that $\lim_t \theta\beta(t)/(\lambda^2 - 1) \mod 1 = \beta_r$ as in Proposition 5, with $r \neq 0$. Then,*

$$\lim_{\inf |t_i - t_j| \to \infty} \phi\Big(W_\theta(f^{(1)})(t_{\nu(1)}) \cdots W_\theta(f^{(j)})(t_{\nu(j)}) W_\theta(g) \\ W_\theta(f^{(j+1)})(t_{\nu(j+1)}) \cdots W_\theta(f^{(n)})(t_{\nu(n)})\Big) = 0 \ ,$$

where $W_\theta(f^{(\ell)})$, $W_\theta(g)$ are defined as in (4.2), by means of finitely supported functions, and

$$W_\theta(f^{(\ell)})(t_{\nu(j)}) = \sum_{\mathbf{m} \in \mathcal{F}_\ell} f^{(\ell)}(\mathbf{m}) W_\theta\Big(T^{t_{\nu(\ell)}} \mathbf{m}\Big)\Big) \ .$$

Proof: Because of linearity and the finiteness of the supports $\mathcal{F}_\ell$ of $f^{(\ell)}$, we can restrict our considerations to expectations of the form

$$\phi\bigg(W_\theta\Big(T^{t_{\nu(1)}}\mathbf{p}^{(1)}\Big) \cdots W_\theta\Big(T^{t_{\nu(j)}}\mathbf{p}^{(j)}\Big) W_\theta(T^s \mathbf{m}) \\ W_\theta\Big(T^{t_{\nu(j+1)}}\mathbf{p}^{(1j+1)}\Big) \cdots W_\theta\Big(T^{t_{\nu(n)}}\mathbf{p}^{(n)}\Big)\bigg) = 0 \ ,$$

where $\mathbf{m} \neq \mathbf{0}$, for the corresponding word is assumed to be centred. If we now group together Weyl operators at equal times, $T^s\mathbf{m}$ is not matched by any of the other operators. On the other hand, using Lemma 2 and considering the smallest difference $|t_{\nu(i)} - t_{\nu(j)}|$ between different times sufficiently large, we can always force upon the supports of Weyl operators at equal times a condition as in (4.20). Otherwise, the expectation would vanish. But then $\mathbf{m} = \mathbf{0}$, which is impossible.

■

The result of above shows that condition (3.8) may hold in systems which are weakly, but not strongly clustering. Therefore, quantized toral automorphisms $(\mathfrak{A}, \Theta, \phi)$ with deformation parameter $\theta \notin \mathcal{Z}$ (see Corollary 4) fall in the class of dynamical systems whose time-asymptotic fluctuations can be handled via Proposition 3, by means of pair partitions only. Condition (3.12) obviously does not hold for any β_r since its equivalent version (4.22) is easily violated. Taking into account that, when $\beta_r \neq 0$, the asymptotic states ϕ_∞ are not

permutation invariant, we can summarize our findings for the previous class of quantized automorphisms of the torus in

Corollary 6. *When the deformation parameter θ is such that $\lim_t \beta(t)\theta/(\lambda^2-1) \mod 1$ equals β_r with β_r as in Proposition 5, the statistics of time-asymptotic fluctuations of quantized hyperbolic automorphisms of the torus $(\mathfrak{A}, \Theta, \phi)$ is as follows. For $\beta_r = 0$, the fluctuations (3.7) of centred observables $W_\theta(f)$ are Gaussian random variables with variance $\sigma = \|f\|_2^2$. For $\beta_r \neq 0$, they obey a distribution law with vanishing odd moments and even moments M_{2n} given by*

$$M_{2n} := \lim_N \phi_\infty\left(F_N\left(W_\theta(f)^{2n}\right)\right) = \frac{1}{n!}\sum_\nu^{(2)} \phi_\infty\left(W_\theta(f)_{\nu(1)} W_\theta(f)_{\nu(2)} \cdots W_\theta(f)_{\nu(2n)}\right)$$

where the sum is over all pair partitions of $\{1, 2, \ldots, n\}$.

Proof: The statement follows from Propositions 3 and 11 and from Corollary 5. ∎

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